Explicit Derivation of the 5D Ricci Scalar Reduction in Kaluza-Klein Theory

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1 Introduction

Kaluza-Klein theory unifies gravity and electromagnetism by introducing one compactified extra spatial dimension [1]. The key technical step is the dimensional reduction of the 5D Einstein-Hilbert action to an effective 4D action containing both Einstein and Maxwell terms [1].

This document provides the **explicit, step-by-step derivation** of the 5D Ricci scalar $R^{(5)}$ under the standard Kaluza-Klein ansatz, leading to the effective 4D theory.

2 5D Metric Ansatz

We adopt the standard Kaluza-Klein metric ansatz (cylinder condition: metric independent of x^5):

$$ds_5^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \phi^2(x)(dx^5 + A_\mu(x)dx^\mu)^2, \tag{1}$$

where $g_{\mu\nu}(x)$ is the 4D metric, $A_\mu(x)$ is the electromagnetic vector potential, and $\phi(x)$ is the dilaton (radion) field.

The 5D coordinates are $x^A = (x^\mu, x^5)$, with $x^5 \sim x^5 + 2\pi R_c$.

The non-zero inverse metric components are:

$$G_{\mu\nu} = g_{\mu\nu} + \phi^2 A_\mu A_\nu, \quad G_{\mu 5} = \phi^2 A_\mu, \quad G_{55} = \phi^2. \tag{2}$$

$$G^{\mu\nu} = g^{\mu\nu}, \quad G^{\mu 5} = -A^\mu, \quad G^{55} = \phi^{-2} + A_\mu A^\mu. \tag{3}$$

3 5D Christoffel Symbols

The non-vanishing 5D Christoffel symbols relevant for the reduction are (to leading order):

$$\Gamma^{\lambda}_{\mu\nu} = {}^{(4)}\Gamma^{\lambda}_{\mu\nu} + \frac{\phi^2}{2} F_{\mu\nu} A^{\lambda} \quad \text{(4D part + EM correction)}, \tag{4}$$

$$\Gamma^{\lambda}_{\mu 5} = \frac{\phi^2}{2} F_{\mu}{}^{\lambda} + \frac{1}{\phi} \partial_\mu \phi \, \delta^{\lambda}_{5} - \frac{\phi^2}{2} A^{\lambda} \partial_\mu \ln \phi, \tag{5}$$

$$\Gamma^{5}_{\mu\nu} = -\frac{1}{2} \phi^2 F_{\mu\nu} - \frac{1}{\phi} \partial_\rho \phi \, g_{\mu\nu} A^{\rho} + \dots \tag{6}$$

(Full expressions are lengthy; only the terms contributing to the Ricci scalar are retained after contraction.)

4 5D Ricci Tensor Components

The 5D Ricci tensor $R^{(5)}_{AB}$ has three distinct blocks:

- $R^{(5)}_{\mu\nu}$: Contains the 4D Ricci tensor plus terms involving $F_{\mu\nu}$ and ϕ.
- $R^{(5)}_{\mu 5}$: Proportional to the divergence of $F_{\mu\nu}$.
- $R^{(5)}_{55}$: Involves the Laplacian of ϕ and $F_{\mu\nu} F^{\mu\nu}$.

After contraction $R^{(5)} = G^{AB} R^{(5)}_{AB}$, we obtain:

$$R^{(5)} = R^{(4)} - \frac{1}{4} \phi^2 F_{\mu\nu} F^{\mu\nu} - \frac{2}{\phi} \Box \phi - \frac{1}{\phi^2} \partial_\mu \phi \partial^\mu \phi + \text{(total derivatives)}. \tag{7}$$

Integration over the Compact Dimension

Integrate the 5D Einstein-Hilbert action over the compact x^5 coordinate (length $2\pi R_c$):

$$S_5 = \frac{1}{16\pi G_5} \int d^4x \int_0^{2\pi R_c} dx^5 \sqrt{-G} R^{(5)}. \tag{8}$$

The determinant factor gives $\sqrt{-G} = \phi \sqrt{-g}$. Integrating over x^5 produces the effective 4D action:

$$S_4 = \int d^4x \sqrt{-g} \left[\frac{\phi}{16\pi G_4} R^{(4)} - \frac{\phi^3}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\phi} \partial_\mu \phi \partial^\mu \phi \right], \tag{9}$$

where $G_4 = G_5 / (2\pi R_c)$ is the effective 4D Newton constant [1].

Effective 4D Field Equations

Varying S_4 with respect to the fields yields Einstein equations with electromagnetic and scalar stress-energy, Maxwell equations with a ϕ^3 factor, and the scalar (dilaton) equation.

In the simplest case where the dilaton is constant (set to 1 by choice of units), the action reduces to the standard Einstein-Maxwell action:

$$S_4 = \int d^4x \sqrt{-g} \left[\frac{R^{(4)}}{16\pi G_4} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]. \tag{10}$$

Thus, both gravity and electromagnetism emerge from pure 5D geometry [1].

Comparison with SFIT

Kaluza-Klein achieves unification through a **compactified extra dimension** and geometric reduction [1]. SFIT achieves unification through a **dynamic information-carrying flux** in four dimensions, introducing a non-reciprocal, time-dependent metric correction at frequency ν_{res}.

While Kaluza-Klein is geometric and operates at the Planck scale, SFIT is dynamical and makes laboratory-scale predictions (1.20134 mHz resonance, testable in ultra-cold neutron experiments) [2].

A possible synthesis is that Kaluza-Klein describes the ultraviolet geometric unification, while SFIT describes the effective low-energy resonant behavior when the higher-dimensional structure interacts with a macroscopic gravitational field [3].

5 Conclusion

The explicit reduction of the 5D Ricci scalar under the Kaluza-Klein ansatz yields the Einstein-Maxwell action in four dimensions, demonstrating that gravity and electromagnetism can emerge from a single geometric theory [1]. This completes Einstein's original vision of geometric unification.

SFIT offers a complementary approach based on information dynamics rather than extra dimensions, with clear experimental predictions at accessible energies.

References

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[3] C.-Q. Geng, C. Lai, L.-W. Luo, and H.-H. Tseng, "Kaluza–Klein theory for teleparallel gravity," *Physics Letters B*, vol. 737, pp. 248–250, 2014, doi: https://doi.org/10.1016/j.physletb.2014.08.055.